Double-Linear Cumulative-Damage Reliability Method

Robert E. Benton, Jr.
Lead Aerospace Engineer
US Army Aviation Engineering Directorate
Redstone Arsenal, AL
robert.e.benton@us.army.mil

ABSTRACT

Fatigue reliability calculations during iterations of the design process have been demonstrated as practical using new analytical and numerical methods based on analytical cumulative-damage reliability tools (a-functions). Effects of fatigue strength and usage distribution may be studied in the presence of load variation. However, to date, the development of the a-functions has been centered on use of the Palmgren-Miner cycle-ratio summation rule for cumulative damage, also known as the Linear Damage Rule (LDR). This paper extends the a-function toolbox for use in determining reliability based on the Manson-Halford Double Linear Damage Rule (DLDR). Examples demonstrate use of the method and charts are provided to illustrate the sensitivity of the cumulative-damage reliability problem to load and strength variations. For a given reliability, probabilistic DLDR methods demonstrate significant reductions in life when compared to LDR methods. The importance of understanding and applying the appropriate LDR, DLDR, or other material characterization is apparent.

NOTATION

A	material constant (log-log, intercept)
$a(\bullet,\bullet)$	Benton a-function defined in [5]
$a^*(\bullet,\bullet,\bullet)$	generalized a-function defined in [7]
$a^*_{DLDRI}(\bullet,\bullet,\bullet,\bullet,\bullet)$	generalized DLDR-based phase I a-function defined herein
$a^*_{DLDRII}(\bullet,\bullet,\bullet,\bullet,\bullet)$	generalized DLDR-based phase II a-function defined herein
B	material constant (log-log, slope)
C	material constant (log-log, cycles at endurance)
COV_E	endurance limit coefficient of variation
COV_S	load amplitude coefficient of variation
D	accumulated fatigue damage
D_{I-II}	accumulated fatigue damage threshold between DLDR phases I and II
D_I	DLDR phase I damage fraction
D_{II}	DLDR phase II damage fraction
E	endurance limit
E_w	working endurance limit, $\mu_E(1 - 3\ COV_E)$
E_δ	critical endurance limit for $D = \delta$
$\exp(\bullet)$	exponential function
$f_E(\bullet)$	endurance limit probability density function
$f_{En}(\bullet)$	E-n joint probability density function
$f_n(\bullet)$	damaging cycles probability density function
$g_E(\bullet)$	damage rate as a function of endurance limit
$g_{En}(\bullet,\bullet)$	accumulated damage as a function of E and n
i	load cycle index
j	component index
k	material constant (factor)
m	material constant (exponent)
m_0	DLDR a- (and b-)function parameter (exponent)
N	number of cycles to failure
N_{ref}	DLDR reference number of cycles
N^*	number of cycles at DLDR break point
N_I	number of cycles within DLDR phase I
N_{II}	number of cycles within DLDR phase II
n	number of applied load cycles
n_I	number of applied phase I load cycles
n_{II}	number of applied phase II load cycles
P	variable used for DLDR (factor)
p	load condition index
Q	variable used for DLDR (exponent)
R	component reliability
S	oscillatory load amplitude
S_{95}	95th percentile oscillatory load amplitude (estimated "top of scatter" load)
v	generalized a-function parameter
x	dummy variable used in Gauss-Laguerre quadrature
y	DLDR a- (and b-)function parameter (factor)
z	standardized normal load amplitude
z_0	standardized normal damaging load amplitude

Presented at the American Helicopter Society 67th Annual Forum, Virginia Beach, VA, May 3-5, 2011.

δ	accumulated fatigue damage at failure, phase I damage fraction at phase I-II threshold, or phase II damage fraction at failure (typically $\delta = 1$)
δ_A	material constant tolerance band (log-log, intercept)
μ	mean (of load amplitudes)
μ_E	mean endurance limit
μ_S	mean load amplitude
ξ	load condition usage spectrum weighting factor
σ	standard deviation (of load amplitudes)
$\phi(\bullet)$	standardized normal probability density function
$\vec{[]}$	vector
$\vec{[]}^T$	vector transpose

INTRODUCTION

Recent advances in helicopter fatigue engineering methods have demonstrated great promise for providing efficient calculations to determine the fatigue reliability of helicopter dynamic components (see [1],[2],[3],[4],[5],[6], and [7] for example). Once mature, such methods will enable helicopter materiel developers and original equipment manufacturers to design more robust equipment while reducing the life-cycle costs inherent to system safety risk management, airworthiness qualification, and continued airworthiness processes.

In [5] and [7], the author has developed and demonstrated new cumulative-damage reliability tools (so called, "a- and b-functions") which may be used to study effects of fatigue strength and usage distribution in the presence of load variation. However, to date, development of a- and b-functions and generalized a- and b-functions has been centered on use of the Palmgren-Miner cycle-ratio summation rule for cumulative damage, also known as the Linear Damage Rule (LDR).

This paper extends generalized a-functions to allow determination of fatigue reliabilities based on an alternative cumulative damage theory, namely the Double Linear Damage Rule (DLDR) developed by Manson and Halford (see [8],[9], and [10] for details). Examples demonstrate use of the DLDR a-functions. In addition, charts are provided to illustrate the sensitivity of the DLDR cumulative-damage reliability problem to load and strength variations.

Readers wishing to understand differences between Palgrem-Miner LDR and Manson-Halford DLDR theories and various other cumulative fatigue theories are referred to the literature survey in [11].

BACKGROUND

ASTM Standard E 739 [12] equation (2a) presents a common characterization of fatigue strength, namely:

$$\log N = A + B\,(\log S) \qquad (1)$$

which is applicable for values of S above the endurance limit. This "log-log" representation of fatigue strength is typically plotted with tolerance bands as:

$$\log N = A + B\,(\log S) \pm \delta_A \qquad (2)$$

Let $N_{EL} = 10^C$ represent the number of cycles at the endurance limit, where $6 \le C \le 10$ and the logarithm of the corresponding endurance limit is

$$\log E = \tfrac{C-A}{B} \qquad (3)$$

As explained in [7], the log-log strength curve may be rearranged to calculate the damage fraction for the i-th cycle applied to the j-th component as

$$\frac{1}{N_{i,j}} = \begin{cases} \frac{1}{k}\left(\frac{S_i}{E_j}\right)^m & if\ S_i > E_j \\ 0 & otherwise \end{cases} \qquad (4)$$

where $k = 10^C$ and $m = -B$ are material constants and $N_{i,j}$ is the number of cycles to failure for a given load S_i and endurance limit E_j. Considering the material constants k and m (and therefore B and C) as invariant within the population of components, the logarithm of the endurance limit for the j-th component is

$$\log E_j = \tfrac{C-A_j}{B} \qquad (5)$$

As derived in [7], the following expression represents the Palmgren-Miner LDR damage accumulated in n cycles for normal-distributed loads:

$$D_j = n\left[\frac{1}{k}\left(\frac{\sigma}{E_j}\right)^m\right] a^*\left(\frac{E_j-\mu}{\sigma}, m, -\frac{\mu}{\sigma}\right) \qquad (6)$$

where μ is the mean load, σ is the load standard deviation, and the "generalized a-function" is defined for $v \le z_0$ as

$$a^*(z_0, m, v) = \int_{z_0}^{\infty} (z-v)^m\,\phi(z)dz \qquad (7)$$

with

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \tag{8}$$

which is the standardized normal probability density function.

Manson and Halford have proposed a Double Linear Damage Rule (DLDR) composed of two phases with distinct linear fatigue damage accumulation patterns. Although the concept of two phases originated from an analogy to crack initiation and crack propagation, these two phases are no longer identified with a definable physical process (see [10]). For the i-th load level and the j-th component, one may describe the number of cycles in each phase as $N_{I,i,j}$ and $N_{II,i,j}$.

Per [10], the fatigue damage accumulated at the end of phase I is

$$D_{I-II} = 0.35 \left(\frac{N_{ref}}{N^*}\right)^{0.25} \tag{9}$$

where N_{ref} is the Palmgren-Miner LDR number of cycles for a reference load level and N^* is the particular life level used to define phase I with a damage-curve break point occurring at damage D_{I-II}, as shown in Figure 1.

Manson and Halford recommend that the value of N_{ref} be selected lower than all life levels in the spectrum. As such, it is typically acceptable to select $N_{ref} = 10^3$ as a lower bound for high-cycle fatigue loads.

Manson and Halford recommend in [8] that N^* be selected as the highest life level in the spectrum and provide an example of minimal effects resulting from alternate N^* selections. In [9] and [10], Manson and Halford propose various more complex spectrum-dependent iterative methods of selecting values for N^*. Notwithstanding such recommendations, the probabilistic nature of the loads and spectrum inherent to present work prompts the author to delay consideration of D_{I-II} spectrum dependency by pursuing initial development of DLDR-based methods based on selecting $N^* = N_{EL} = 10^C$ (*i.e.*, the highest *damaging* life level possible in any spectrum). Once the author's DLDR methods are established, readers may wish to use the methods derived herein to assess the potential benefits of re-introducing D_{I-II} spectrum dependency into more complex methods.

Adapting Manson-Halford equations in [8] to avoid notation conflicts and facilitate calculating damage

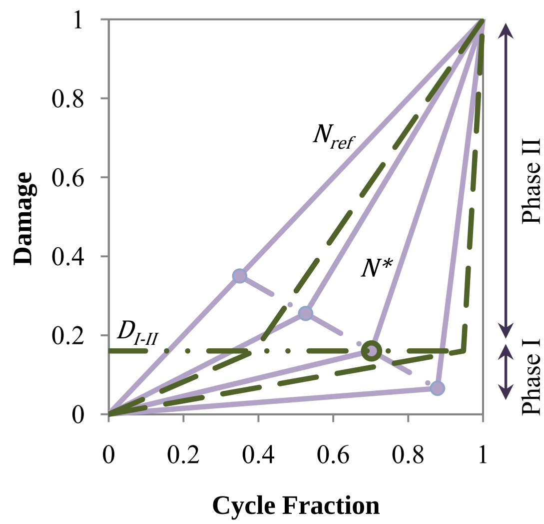

Figure 1: Double Linear Damage Rule Determination of DLDR phases using the N^* damage-curve break point (adapted from [10]).

for the i-th load level and the j-th component, the number of cycles in phases I and II are calculated as

$$N_{I,i,j} = N_{i,j} \exp\left(P {N_{i,j}}^{Q}\right) \tag{10}$$

and

$$N_{II,i,j} = N_{i,j} - N_{I,i,j} \tag{11}$$

respectively, where intermediate variables Q and P are defined as

$$Q = \frac{1}{\ln\left(\frac{N_{ref}}{N^*}\right)} \ln\left[\frac{\ln(D_{I-II})}{\ln\left(1 - \frac{0.65}{0.35} D_{I-II}\right)}\right] \tag{12}$$

$$P = \frac{\ln D_{I-II}}{{N_{ref}}^{Q}} \tag{13}$$

and where $N_{i,j}$ is the LDR number of cycles to failure for the i-th load level and the j-th component.

Dashed lines in Figure 1 show the results of adjusting damage curve break points for life levels other than N_{ref} and N^* to match the $N_{I,i,j} = N_{i,j} \exp\left(P {N_{i,j}}^{Q}\right)$ equation based the N^* break point. Thus, the DLDR formulation allows fatigue damage to be considered linear in each phase while closely approximating damage curves generated from two-load-level fatigue test results (see [8], [9], and [10] for these test results and other details related to the DLDR formulation).

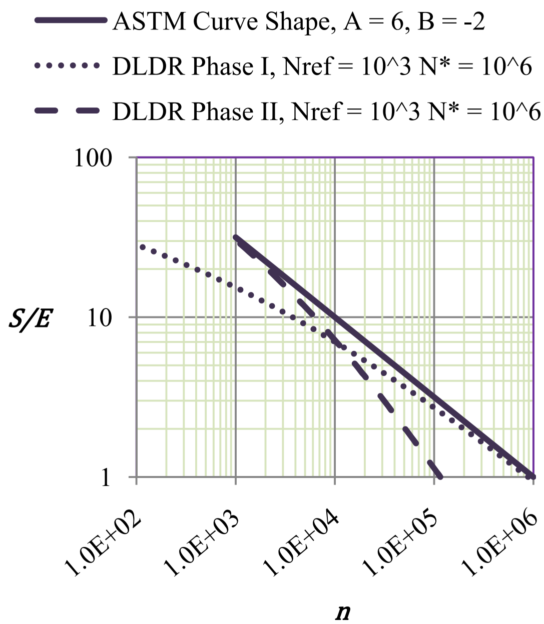

Figure 2: Comparison of the number of cycles in DLDR phases I and II to LDR cycles to failure – total DLDR phase I and II cycles at any given load level equals LDR cycles but high values of S/E in phase I limit lower values in phase II.

Figure 2 further illustrates the DLDR concept where high values of S/E during phase I would limit the number of cycles to failure for lower values of S/E during phase II. However, for simple cases involving a single load level, the number of cycles is split between the two phases and the total number of cycles remains unchanged.

DOUBLE-LINEAR RELIABILITY THEORY

This paper aims to establish a double-linear reliability theory based on expressions using generalized a-functions or similar for each DLDR phase. The inverse of phase I cycles for the i-th cycle applied to the j-th component may be expressed as

$$\frac{1}{N_{I,i,j}} = \begin{cases} \dfrac{\frac{1}{k}\left(\frac{S_i}{E_j}\right)^m}{\exp\left\{P\left[\frac{1}{k}\left(\frac{S_i}{E_j}\right)^m\right]^{-Q}\right\}} & if\ S_i > E_j \\ 0 & otherwise \end{cases} \tag{14}$$

for which generalized a-functions are not applicable without first incorporating the effects of the denominator terms absent from similar expressions for LDR. As such, a new class of generalized DLDR-based phase I a-functions is defined herein for $v < z_0$ as

$$a^*_{DLDRI}(z_0, m, v, y, m_0) = \int_{z_0}^{\infty} \frac{(z-v)^m}{\exp[y(z-v)^{m_0}]}\,\phi(z)dz \tag{15}$$

where $\phi(z)$ is the standardized normal probability density function.

Generalized DLDR-based phase I a-functions may be applied to phase I damage calculations, as follows

$$D_{I,j} = n_I\, g_{E,I}(E_j) \tag{16}$$

where $D_{I,j}$ is the Manson-Halford DLDR phase I damage fraction and the corresponding phase I damage rate $g_{E,I}(E_j)$ is calculated for the case of normal-distributed loads as

$$g_{E,I}(E_j) = \left[\frac{1}{k}\left(\frac{\sigma}{E_j}\right)^m\right] \times a^*_{DLDRI}\left(\frac{E_j-\mu}{\sigma}, m, -\frac{\mu}{\sigma}, Pk^Q\left(\frac{\sigma}{E_j}\right)^{m_0}, m_0\right) \tag{17}$$

where $m_0 = -mQ$. In accordance with Manson-Halford DLDR theory, the j-th component is predicted to enter phase II after accumulating phase I damage of $D_{I,j} = 1$.

The inverse of phase II cycles for the i-th cycle applied to the j-th component may be expressed as

$$\frac{1}{N_{II,i,j}} = \begin{cases} \dfrac{\frac{1}{k}\left(\frac{S_i}{E_j}\right)^m}{1-\exp\left\{P\left[\frac{1}{k}\left(\frac{S_i}{E_j}\right)^m\right]^{-Q}\right\}} & if\ S_i > E_j \\ 0 & otherwise \end{cases} \tag{18}$$

for which a new class of generalized DLDR-based phase II a-functions is defined herein for $v < z_0$ as

$$a^*_{DLDRII}(z_0, m, v, y, m_0) = \int_{z_0}^{\infty} \frac{(z-v)^m}{1-\exp[y(z-v)^{m_0}]}\,\phi(z)dz \tag{19}$$

As in the case of phase I, generalized DLDR-based phase II a-functions may be applied to phase II damage calculations, as follows

$$D_{II,j} = n_{II}\, g_{E,II}(E_j) \tag{20}$$

where $D_{II,j}$ is the Manson-Halford DLDR phase II damage fraction and the corresponding phase II damage rate $g_{E,II}(E_j)$ is calculated for the case of normal-distributed loads as

$$g_{E,II}(E_j) = \left[\frac{1}{k}\left(\frac{\sigma}{E_j}\right)^m\right] \times a^*_{DLDRII}\left(\frac{E_j - \mu}{\sigma}, m, -\frac{\mu}{\sigma}, Pk^Q\left(\frac{\sigma}{E_j}\right)^{m_0}, m_0\right) \qquad (21)$$

Per Manson-Halford DLDR theory, the j-th component is predicted to fail after entering phase II and accumulating phase II damage of $D_{II,j} = 1$.

For completeness, similar DLDR-based functions for use with Weibull loads (extending the b-function tools from [5] and [7]) are developed in Appendix A.

EVALUATION TECHNIQUES

Although the author introduced analytic evaluation of a certain subset of a-functions in [5], the author recommends numerical integration techniques introduced in [7] be applied for evaluation of the generalized DLDR a-functions, $a^*_{DLDRI}(\bullet,\bullet,\bullet,\bullet,\bullet)$ and $a^*_{DLDRII}(\bullet,\bullet,\bullet,\bullet,\bullet)$.

In [7], evaluation of a-functions with non-integer m values is accomplished using 15-point Gauss-Laguerre quadrature formulas (see [13] and [14] for details). To correct for inaccuracies in use of Gauss-Laguerre for evaluating the a-function for small values of $|z_0|$ and m, [7] recommends that the a-function integral be evaluated in two steps. Adapting the methods of [7] to the generalized DLDR phase I a-functions integral results in the following:

$$a^*_{DLDRI}(z_0, m, v, y, m_0) = a_1 + a_2 \qquad (22)$$

where

$$a_1 = \int_{z_0}^{z_1} \frac{(z - v)^m}{\exp[y(z - v)^{m_0}]}\, \phi(z)dz \qquad (23)$$

$$a_2 = \int_{z_1}^{\infty} \frac{(z - v)^m}{\exp[y(z - v)^{m_0}]}\, \phi(z)dz \qquad (24)$$

and

$$z_1 = 1 + \tfrac{1}{2}z_0 + \sqrt{m + \tfrac{1}{4}{z_0}^2} \qquad (25)$$

It is recommended that the integral for $a_1$be evaluated using a Newton-Coats formula, such as a 1000-step 4th-order formula (*i.e.*, Boole's method, see [13] for details).

Implementation of Gauss-Laguerre requires a change of coordinates. As was the case in [7], let $x = \frac{1}{2}z^2$ and $x_{shift} = \frac{1}{2}{z_1}^2$ for the generalized DLDR a-function. One may evaluate a_2 as

$$a_2 = \int_{x_{shift}}^{\infty} f(x)\exp(-x)\,dx \approx \exp(-x_{shift})\sum_{i=1}^{n} w_i\, f(x_i + x_{shift}) \qquad (26)$$

where

$$f(x) = \frac{1}{\sqrt{4\pi x}}\ \frac{\left(\sqrt{2x} - v\right)^m}{\exp\left[y\left(\sqrt{2x} - v\right)^{m_0}\right]} \qquad (27)$$

and values for w_i and x_i are provided in [14].

Evaluation of generalized DLDR phase II a-functions, $a^*_{DLDRII}(\bullet,\bullet,\bullet,\bullet,\bullet)$ may be accomplished with similar methods after making the following modifications to certain of the above expressions:

$$a^*_{DLDRII}(z_0, m, v, y, m_0) = a_1 + a_2 \qquad (28)$$

$$a_1 = \int_{z_0}^{z_1} \frac{(z - v)^m}{1 - \exp[y(z - v)^{m_0}]}\, \phi(z)dz \qquad (29)$$

$$a_2 = \int_{z_1}^{\infty} \frac{(z - v)^m}{1 - \exp[y(z - v)^{m_0}]}\, \phi(z)dz \qquad (30)$$

$$f(x) = \frac{1}{\sqrt{4\pi x}}\ \frac{\left(\sqrt{2x} - v\right)^m}{1 - \exp\left[y\left(\sqrt{2x} - v\right)^{m_0}\right]} \qquad (31)$$

Numerical procedures and all other expressions remain unchanged from those used in evaluation of the generalized DLDR phase I a-function.

APPLICATION NOTES

In this section, generalized DLDR-based phase I and phase II a-functions are applied to three special cases of fatigue reliability problems. Each case adapts results in [7] to study double-linear reliability in the presence of load and strength variation. Two cases of deterministic usage are considered, one for usage characterized by a single load condition and the second for a distributed usage over multiple load conditions. Also, a probabilistic usage case is considered for a single load condition.

An approach to apply LDR-based a-function tools to multivariate probabilistic usage problems is presented in [7]. However, due to the complexity of the effective composite DLDR damage rate for multiple load conditions, a similar approach is not available for application to the DLDR case at this time.

Single Load Condition Deterministic Usage

Application is first considered for deterministic usage with a single load condition. At component failure, accumulated damage D_j in the j-th component is equal to some threshold value, δ (typically failure is considered to occur at $\delta = 1$). At failure, the phase I and phase II damage fractions are also at the threshold, namely: $D_{I,j} = \delta$ and $D_{II,j} = \delta$.

The reader may object to the concept of using the same threshold value for phase I and II. In fact, the sequence of damage accumulation is that phase I is first accomplished prior to initiation and completion of phase II. However, the author's concept of the threshold is such that both phases have been completed at failure by reaching the threshold (each in turn). Sequence is only important if the deterministic usage changes with time and varies between phase I and phase II. In such cases, the phase I and II damage rates should be calculated according to the appropriate usage spectrum during each phase and the following discussion would remain applicable.

The number of cycles to failure (specifically, to achieve $D_j = \delta$) is calculated as

$$n = n_I + n_{II} = \delta\left(\frac{1}{g_{E,I}(E_j)} + \frac{1}{g_{E,II}(E_j)}\right) \tag{32}$$

Solving for δ yields the accumulated damage as

$$D_j = g_{En}(E_j, n) = n\, g_E(E_j) \tag{33}$$

where

$$g_E(E_j) = \frac{g_{E,I}(E_j)g_{E,II}(E_j)}{g_{E,I}(E_j) + g_{E,II}(E_j)} \tag{34}$$

may be considered the effective composite DLDR damage rate.

Let E be randomly distributed with density function $f_E(E)$. The probability that the damage in a given component is less than the failure threshold δ is

$$\Pr(D < \delta) = F_D(\delta) = \int_{\substack{\text{All values}\\ \text{of E where}\\ g_{En}(E,n)<\delta}} f_E(E)\, dE \tag{35}$$

where $\Pr(D < \delta)$ is the component's reliability, R.

Conversely, the probability that the damage in a given component exceeds the failure threshold δ is

$$\Pr(D > \delta) = \int_{\substack{\text{All values}\\ \text{of } E \text{ where}\\ g_{En}(E,n)>\delta}} f_E(E)\, dE \tag{36}$$

where $\Pr(D > \delta)$ is the probability of component failure. Considering a specific critical endurance limit E_δ such that $g_{En}(E_\delta, n) = \delta$ for a given value of n, one may clarify the integration as

$$\Pr(D > \delta) = \int_{-\infty}^{E_\delta} f_E(E)\, dE \tag{37}$$

which is

$$\Pr(D > \delta) = F_E(E_\delta) \tag{38}$$

by definition, where $F_E(\bullet)$ is the cumulative endurance limit distribution. It is important to clarify that the particular type of fatigue strength distribution is not specified (*i.e.*, the methods of this paper are applicable to normal strength, log-normal strength, Weibull strength, *etc.*).

Single Load Condition Probabilistic Usage

With probabilistic usage, the number of cycles of each load condition at retirement is also a random variable. The simple case of a single damaging load condition is considered, where the accumulated damage may be expressed as

$$D_j = n_j g_E(E_j) = g_{En}(E_j, n_j) \tag{39}$$

where

$$g_E(E_j) = \frac{g_{E,I}(E_j)g_{E,II}(E_j)}{g_{E,I}(E_j) + g_{E,II}(E_j)} \tag{40}$$

is the effective composite DLDR damage rate.

For this case, one may substitute the effective composite DLDR damage rate directly into the discussion of probabilistic usage in [7], where the probability that the damage is above some threshold δ (probability of component failure) is calculated as

$$\Pr(D > \delta) = F_E(0) + \int_0^\infty f_E(E) \left[1 - F_n\left(\frac{\delta}{g_E(E)}\right)\right] dE \tag{41}$$

where E and n are independent and randomly distributed with density functions $f_E(E)$ and $f_n(n)$ and cumulative distributions $F_E(E)$ and $F_n(n)$, respectively.

Multiple Load Conditions Deterministic Usage

For the case of deterministic usage with p multiple (potentially damaging) load conditions in the usage spectrum, the Manson-Halford DLDR phase I and phase II damage fractions are calculated as follows:

$$D_{I,j} = n_I\, \vec{\xi}^T\, \overrightarrow{g_{E,I}}(E_j) \tag{42}$$

and

$$D_{II,j} = n_{II}\, \vec{\xi}^T\, \overrightarrow{g_{E,II}}(E_j) \tag{43}$$

respectively, where

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{bmatrix} \tag{44}$$

is a vector of load condition weighting factors describing the usage spectrum, and the vectors $\overrightarrow{g_{E,I}}(E_j)$ and $\overrightarrow{g_{E,II}}(E_j)$ contain corresponding DLDR phase I and phase II damage rates for each load condition in the usage spectrum.

The number of cycles to failure ($D_j = \delta$) is

$$n = n_I + n_{II} = \delta\left(\frac{1}{\{\vec{\xi}^T\, \overrightarrow{g_{E,I}}(E_j)\}} + \frac{1}{\{\vec{\xi}^T\, \overrightarrow{g_{E,II}}(E_j)\}}\right) \tag{45}$$

Solving for δ yields the accumulated damage as

$$D_j = g_{En}(E_j, n) = n\, g_E(E_j) \tag{46}$$

where

$$g_E(E_j) = \frac{\{\vec{\xi}^T\, \overrightarrow{g_{E,I}}(E_j)\}\{\vec{\xi}^T\, \overrightarrow{g_{E,II}}(E_j)\}}{\{\vec{\xi}^T\, \overrightarrow{g_{E,I}}(E_j)\} + \{\vec{\xi}^T\, \overrightarrow{g_{E,II}}(E_j)\}} \tag{47}$$

is the effective composite DLDR damage rate for multiple load conditions.

As before, one may consider a specific critical endurance limit E_δ such that $g_{En}(E_\delta, n) = \delta$ for a given value of n and solve for the probability of component failure as

$$\Pr(D > \delta) = F_E(E_\delta) \tag{48}$$

where $F_E(\bullet)$ is the cumulative strength distribution.

EXAMPLES

Examples are used demonstrate application to single and multiple load cases.

Tables 1 through 4 present an example intended to allow demonstration of the double linear reliability methods developed in this paper. Table 1 presents the problem parameters where k and m correspond to the curve shape shown in Figure 2 and the mean load is twice the mean endurance limit.

For those unfamiliar with a-functions presented in [5] and [7], a baseline solution is developed via application of traditional methods with results provided in Table 2. A fatigue life of just over 90,000 cycles is calculated using a fatigue curve with mean minus three sigma working endurance limit, E_w, and a load level corresponding to 95th percentile (or top of scatter) vibratory loads, S_{95}.

Table 3 presents results for the example problem using double linear theories developed in this paper. A fatigue life of nearly 68,000 cycles corresponds to "six nines" of reliability ($R = 0.999999$) for the problem characterized by parameters in Table 1, which is 25% less than the fatigue life predicted using traditional methods (see Table 2). The reader should attempt replicating the single load case results provided in Table 3 prior to attempting more realistic problems involving multiple-load cases.

Table 4 presents results for LDR methods developed in [7]. These results are presented to allow the reader to test the ability to duplicate simpler LDR results prior to attempting DLDR methods. As one might expect, comparison of DLDR and LDR results in Tables 3 and 4, respectively, indicates little difference between DLDR-based and LDR-based reliability methods when applied to problems with a single damaging load condition.

Table 1: Parameters Used for a Single Load Condition Example

Parameter	Value
k	1.0×10^6
m	2
μ_E	5
COV_E	0.1
μ_S	10
COV_S	0.1
N_{ref}	1.0×10^3
N^*	1.0×10^6

Table 2: Results for Traditional LDR Method[†]

Variable Name	Value[‡]
E_w	3.50000
S_{95}	11.6449
n (per traditional method)	90337

[†] Calculation is based on mean minus 3 sigma working fatigue strength curve and 95-th percentile top of scatter vibratory load.
[‡] Note: intermediate results in this and each of the other tables in this paper are rounded to 6 digits and calculated fatigue lives are rounded to the nearest whole cycle.

Table 3: Single Load Condition Example Results with a Target Reliability of "Six Nines" (0.999999)

Variable Name	Value
D_{I-II}	0.0622398
P	-62.7717
Q	-0.451408
E_{crit}	2.62329
z_0	-7.37671
v	-10.0000
y	-0.0514247
m_0	0.902817
$a^*_{DLDRI}(z_0, m, v, y, m_0)$	153.562
$a^*_{DLDRII}(z_0, m, v, y, m_0)$	297.367
$g_{E,I}$	2.23147×10^{-5}
$g_{E,II}$	4.32117×10^{-5}
g_E	1.47155×10^{-5}
n	67955

Table 4: Comparative Results for LDR Case with Six Nines Target Reliability

Variable Name[§]	Value
$a^*(z_0, m, v)$	101.000
g_E	1.46767×10^{-5}
n	68135

[§] Note, variables in this table are defined per methods presented in [7] rather than double linear equations herein.

To demonstrate application to problems involving multiple load conditions, an example is borrowed from [9] where four load conditions are applied which would produce nominal lives of 10^3, 10^4, 10^5, and 10^6 cycles (for mean loads and strength). As in [9], each block of loads contains 1 percent of the nominal failure cycles for each loading level individually. Tables 5 and 6 adapt this example to demonstrate the methods developed in this paper. It is noted that the author has modified the problem by assuming that each "block" of loads corresponds to 400 flight hours (selected to correspond to 100 hours per damaging load condition).

Results are presented in Figure 3, where four solutions are shown for each of the LDR and the DLDR theories:

- Nominal method (mean endurance limit)
- Traditional method (working endurance limit)
- Probabilistic method (a-functions)
- Probabilistic method (a-functions) with additional load below working endurance limit

First, a life is calculated using nominal methods applied for the mean endurance limit (100 load units), resulting in a 10,000 hour life for LDR and a 4,596 hour life for DLDR. The LDR result is as expected when considering that four load conditions are applied per 400 hour block which each produce 1% damage fraction per block (4% damage per 400 hours ratios to 100% damage per 10,000 hours). Also, the DLDR result corresponds to within 0.2% of that presented in [9] (after adjusting for 400 hours per load block).

Second, using the working endurance limit (70 load units), traditional methods result in lives of 4,900 hours and 2,089 hours for LDR and DLDR cases, respectively.

Next, the methods provided in [7] and herein are applied to the problem based on the strength Coefficient of Variation (COV) and the load COV specified in Table 5. To maintain six nines of reliability, the resulting lives are 2,725 hours and 1,222 hours for LDR and DLDR cases, respectively.

Finally, the usage spectrum in Table 7 is considered, where an additional load is included with load level below the working endurance limit, E_w, and occurring 10,000 cycles per 400 hour block. This level of load would be neglected by traditional methodologies where loads below E_w do not impact the calculation. However, the probabilistic methods provided in [7] and herein are impacted at high

Table 5: Parameters Used for a Multiple Load Condition Example

Parameter	Value
k	1.0×10^6
m	2
μ_E	100
COV_E	0.1
μ_S	varies
COV_S	0.1
N_{ref}	1.0×10^3
N^*	1.0×10^6

Table 6: Usage Spectrum for Example with Multiple Load Conditions

Index, p	Mean Load Level	Nominal Cycles to Failure	Cycles per 400 hour Block
1	3162.28	10^3	10
2	1000.00	10^4	100
3	316.228	10^5	1000
4	100.000	10^6	10000

Table 7: Alternative Usage Spectrum for Example with Multiple Load Conditions

Index, p	Mean Load Level	Nominal Cycles to Failure	Cycles per 400 hour Block
1	3162.28	10^3	10
2	1000.00	10^4	100
3	316.228	10^5	1000
4	100.000	10^6	10000
5	66.6667	∞	10000

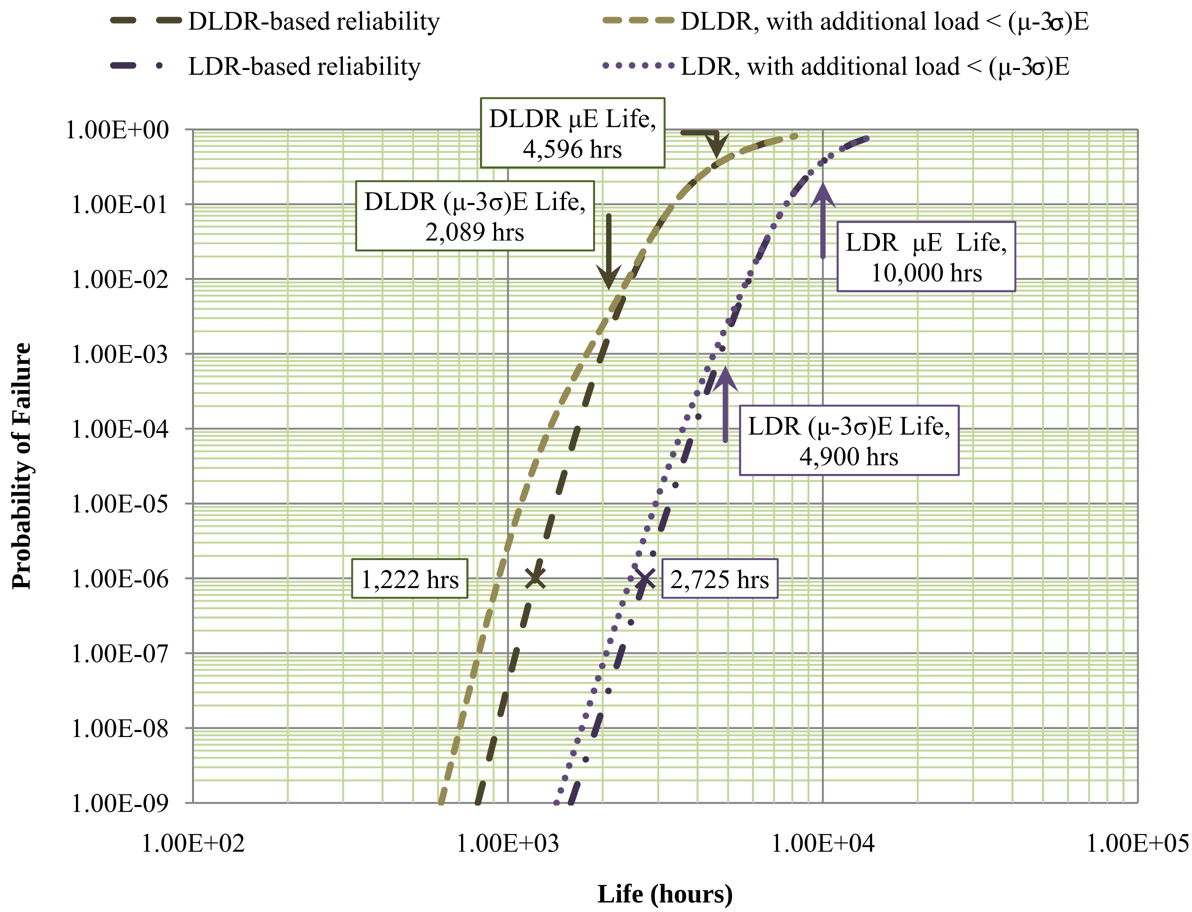

Figure 3: Results for the Multiple-Load-Condition Example (Adapted from [9]) with 400 Hour Load Blocks

reliability levels based on overlap of the load and strength distribution tails. To maintain six nines of reliability, the resulting lives are 2,455 hours and 934 hours for LDR and DLDR cases, respectively. The impact of the additional load level below E_w is magnified in the DLDR case due to reductions in cycles to failure at low load-levels during phase II (see Figure 2).

SENSITIVITY STUDIES

This section illustrates the sensitivity of the DLDR cumulative-damage reliability tools to load and strength variations. To avoid complex sensitivity studies based on multiple load condition interactions, the single load condition example with parameter values provided in Table 1 is taken as a baseline for comparison. Although single load condition examples do not take full advantage of DLDR theories, these sensitivity studies fully apply the DLDR method and may be used to understand the impact of changes in any given load condition.

Figures 4 and 5 demonstrate the increase in probability of failure (decrease in reliability) as the number of cycles is increased and with increasing load ratio. Although the nominal ASTM log-log strength curve demonstrates a sharp corner at $N_{EL} = 10^C$, the author notes that probabilistic methods tend to smooth the corner.

Figure 6 presents results for increases in load COV with six nines reliability. Load variability effects are more significant below the endurance limit where the curves in Figure 6 “fan out” with additional cycles and increased load variation. Figure 6 demonstrates the importance of understanding load distributions for sustained steady-state conditions such as level flight, which are typically underrepresented in a composite worst case usage spectrum. Figure 7 presents results for increases in strength COV. Comparison of the 14% strength COV case from Figure 7 to the 50% load COV case from Figure 6 indicates greater sensitivity to strength variation at each load level (see [5] for similar LDR results).

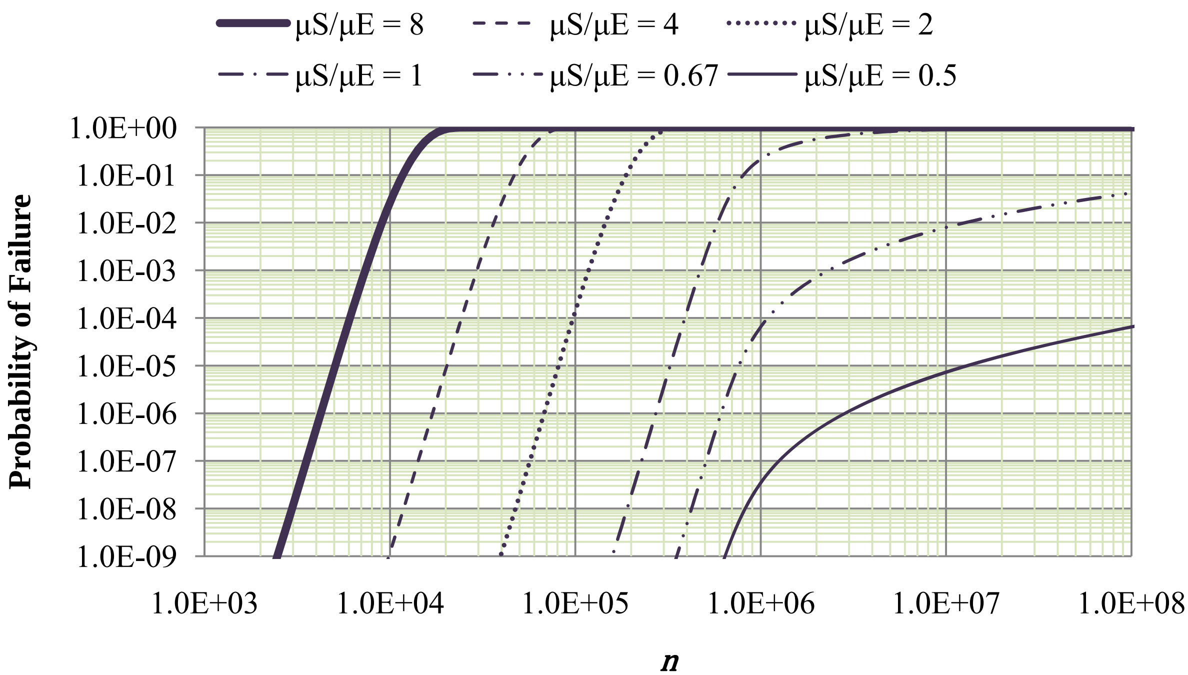

Figure 4: Probability of Failure as a Function of Cycles for Various Ratios of Mean Load to Mean Strength, μ_S/μ_E, with $k = 10^6$, $m = 2$, $COV_E = 0.1$, $COV_S = 0.1$, $N_{ref} = 10^3$, and $N^* = 10^6$

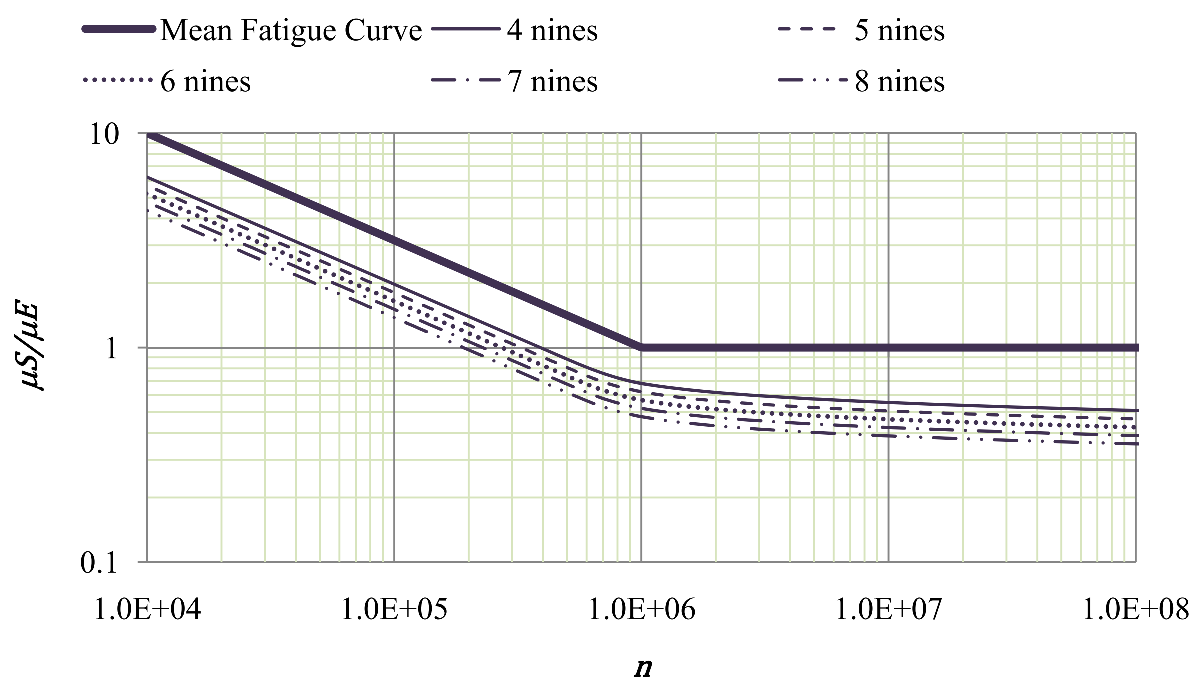

Figure 5: Ratio of Mean Load to Mean Strength, μ_S/μ_E, as a Function of Cycles for Various Reliability Levels, with $k = 10^6$, $m = 2$, $COV_E = 0.1$, $COV_S = 0.1$, $N_{ref} = 10^3$, and $N^* = 10^6$

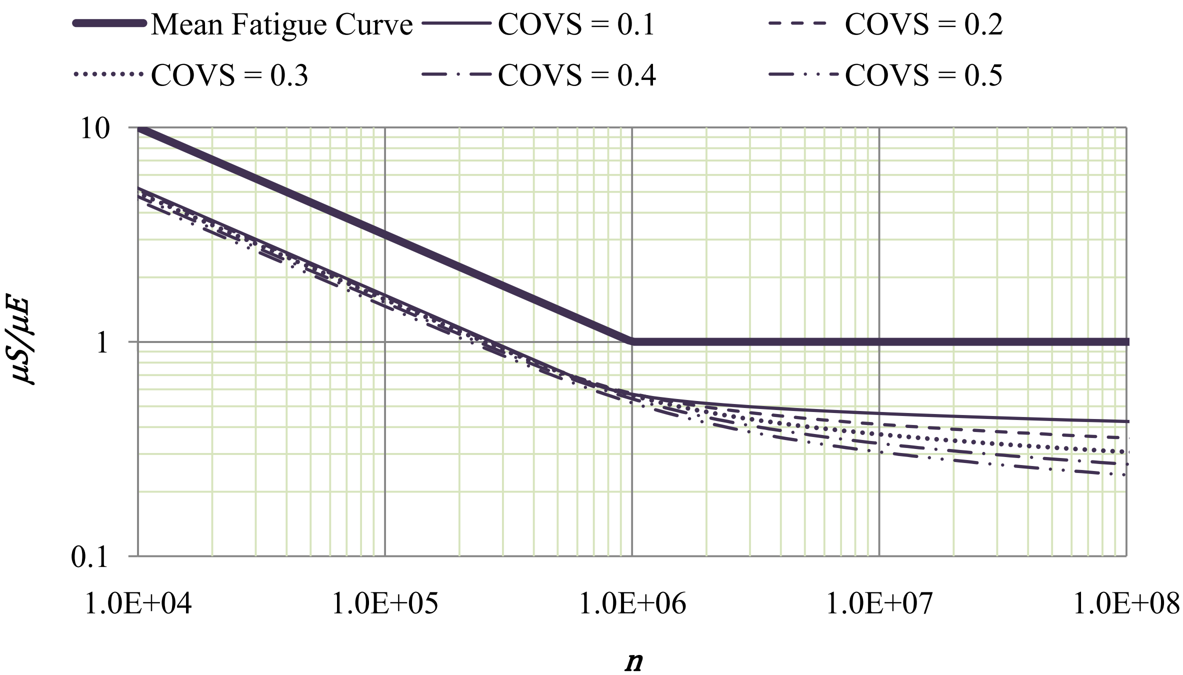

Figure 6: Ratio of Mean Load to Mean Strength, μ_S/μ_E, as a Function of Cycles for Various Levels of Load Variation, with six nines reliability, $k = 10^6$, $m = 2$, $COV_E = 0.1$, $N_{ref} = 10^3$, and $N^* = 10^6$

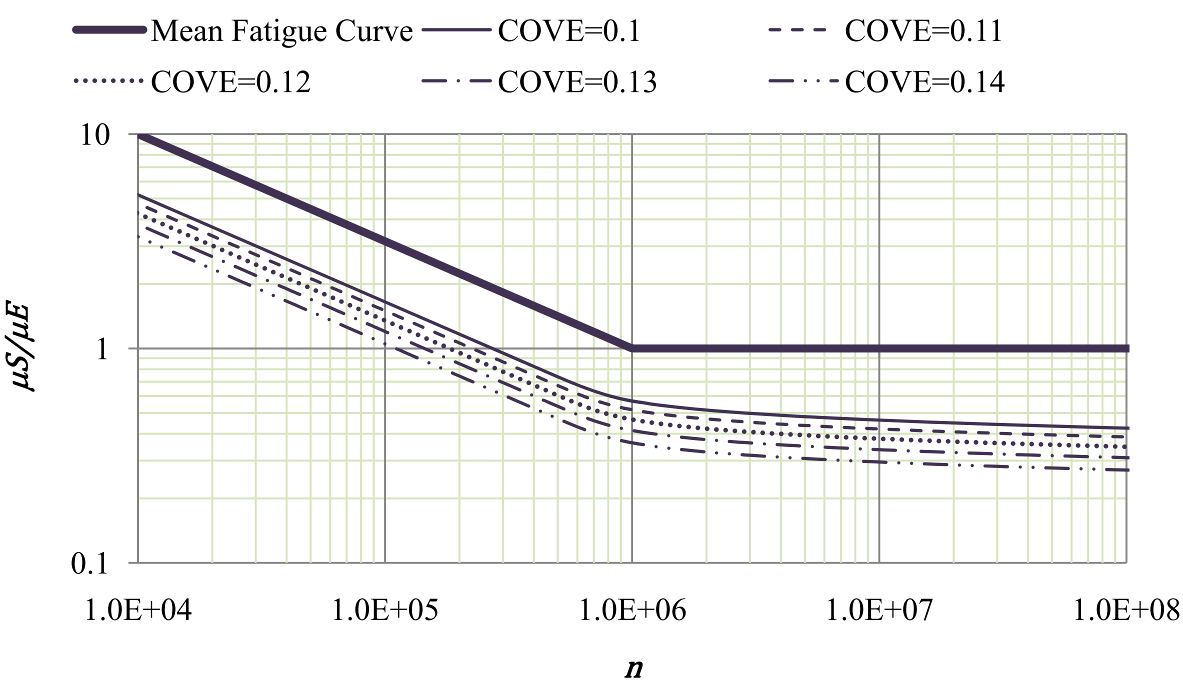

Figure 7: Ratio of Mean Load to Mean Strength, μ_S/μ_E, as a Function of Cycles for Various Levels Strength Variation, with six nines reliability, $k = 10^6$, $m = 2$, $COV_S = 0.1$, $N_{ref} = 10^3$, and $N^* = 10^6$

CONCLUSIONS

The author's conclusions are as follows:

1. The work described in this paper extends the generalized α-function tools developed in [5] and [7] to allow calculation of fatigue reliability based on the Manson-Halford DLDR, thus formulating a new double-linear reliability theory.
2. A sensitivity study has been performed which indicates the importance of understanding the strength distribution, especially the strength COV.
3. A multiple-load-condition example clearly demonstrates that the selection of cumulative damage theory impacts reliability.
   a. Probabilistic DLDR methods demonstrate significant reductions in fatigue life to maintain six nines of reliability when compared to LDR methods.
   b. The importance of understanding and applying the appropriate LDR, DLDR, or other material characterization is apparent.
4. Future efforts should include the following:
   a. The potential benefits of re-introducing D_{I-II} spectrum dependency into more complex methods should be assessed.
   b. The applicability of double-linear reliability methods described in this paper would be significantly improved by the development of an approach to apply DLDR-based methods to multivariate probabilistic usage problems.

ACKNOWLEDGEMENTS

The author wishes to thank co-workers who have reviewed and suggested improvements to various drafts of this paper.

REFERENCES

[1] Michael A. Brown and Jung-Hua Chang, "Analytical Techniques for Helicopter Component Reliability", *American Helicopter Society 64th Annual Forum Proceedings*, Montreal, Canada, April 29-May 1, 2008.

[2] D.O. Adams and J. Zhou, "Searching for the Usage Monitor Reliability Factor Using an Advanced Fatigue Reliability Assessment Model", *American Helicopter Society 65th Annual Forum Proceedings*, Grapevine, TX, May 27-29, 2009.

[3] John M. McFarland and David S. Riha, "Uncertainty Quantification Methods for Helicopter Fatigue Reliability Analysis", *American Helicopter Society 65th Annual Forum Proceedings*, Grapevine, TX, May 27-29, 2009.

[4] Terry J. Larchuk, "Fatigue Load Distribution and Variability Modeling", *American Helicopter Society 64th Annual Forum Proceedings*, Montreal, Canada, April 29-May 1, 2008.

[5] Robert E. Benton, Jr., "Cumulative-Damage Reliability for Random-Independent (Normal- or Weibull-Distributed) Fatigue Stress, Random-Fixed Strength, and Deterministic Usage", *American Helicopter Society 65th Annual Forum Proceedings*, Grapevine, TX, May 27-29, 2009.

[6] J. Zhao and D. O. Adams, "Achieving Six-Nine's Reliability Using an Advanced Fatigue Reliability Assessment Model", *American Helicopter Society 66th Annual Forum Proceedings*, Phoenix, AZ, May 11-13, 2010.

[7] Robert E. Benton, Jr., "Further Advances in a Recently Developed Cumulative-Damage Reliability Method", *American Helicopter Society 66th Annual Forum Proceedings*, Phoenix, AZ, May 11-13, 2010.

[8] S. S. Manson and G. R. Halford, *Practical Implementation of the Double Linear Damage Rule and Damage Curve Approach for Treating Cumulative Fatigue Damage*, NASA Technical Memorandum NASA-TM-81517, April 1980.

[9] S. S. Manson and G. R. Halford, *Re-examination of Cumulative Fatigue Damage Analysis—An Engineering Perspective*, NASA Technical Memorandum NASA-TM-87325, July 1985.

[10] S. S. Manson and G. R. Halford, *Fatigue and Durability of Structural Materials*, ASM International, Materials Park, Ohio, 2006.

[11] A. Fatemi and L. Yang, "Cumulative fatigue damage and life prediction theories: a survey of the state of the art for homogeneous materials", *Int. J. Fatigue*, Vol. 20, No. 1, pp. 9-34, 1998.

[12] ASTM Standard Practice E 739 – 91, *Standard Practice for Statistical Analysis of Linear or Linearized Stress-Life (S-N) and Strain Life (ϵ-N) Fatigue Data*, 1991.

[13] Singiresu S. Rao, *Applied Numerical Methods for Engineers and Scientists*, Prentice Hall, Upper Saddle River, NJ, 2002.

[14] Milton Abramowitz and Irene A. Stegun, eds., *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series 55, National Bureau of Standards, Washington, D.C., June 1964.

APPENDIX A

This appendix presents double-linear reliability theory for use with Weibull distributed loads by extending the b-function methods of [5] and [7]. Although the material in this appendix is not required to understand and apply the a-function methods to problems involving normally distributed loads, these tools are provided for completeness.

Additional Notation used in Appendix A:

$b(\bullet,\bullet,\bullet)$	Benton b-function defined in [5]
$b^*(\bullet,\bullet,\bullet,\bullet)$	generalized b-function defined in [7]
$b^*_{DLDRI}(\bullet,\bullet,\bullet,\bullet,\bullet,\bullet)$	generalized DLDR-based phase I b-function defined herein
$b^*_{DLDRII}(\bullet,\bullet,\bullet,\bullet,\bullet,\bullet)$	generalized DLDR-based phase II b-function defined herein
u	generalized b-function parameter
w	"standardized" Weibull load amplitude
w_i	weighting value used in Gauss-Laguerre quadrature
w_0	"standardized" Weibull damaging load amplitude
β	Weibull slope (for load amplitudes)
$\gamma(\bullet,\bullet)$	"standardized" Weibull probability density function
δ	Weibull minimum expected value (for load amplitudes) – *also used above as accumulated fatigue damage*
η	Weibull characteristic value (for load amplitudes)

Double-Linear Reliability Theory for Weibull Loads

The DLDR phase I damage rate, $g_{E,I}(E_j)$, is calculated for the case of Weibull loads as

$$g_{E,I}(E_j) = \left[\frac{1}{k}\left(\frac{\eta-\delta}{E_j}\right)^m\right] \times b^*_{DLDRI}\left(\frac{E_j-\delta}{\eta-\delta}, m, \beta, -\frac{\delta}{\eta-\delta}, Pk^Q\left(\frac{\eta-\delta}{E_j}\right)^{m_0}, m_0\right) \tag{A-1}$$

where and P, k, Q, and m_0 are as defined in the main body of this paper; β, η, and, δ are Weibull distribution parameters; and the new class of generalized DLDR-based phase I b-functions is defined for $u \le \max(w_0, 0)$ as

$$b^*_{DLDRI}(w_0, m, \beta, u, y, m_0) = \int_{\max(w_0,0)}^{\infty} \frac{(w-u)^m}{\exp[y(w-u)^{m_0}]}\,\gamma(w,\beta)dw \tag{A-2}$$

with

$$\gamma(w,\beta) = \begin{cases} \beta\, w^{\beta-1}\exp\left(-w^{\beta}\right) & if\ w \ge 0 \\ 0 & otherwise \end{cases} \tag{A-3}$$

Similarly, the DLDR phase II damage rate, $g_{E,II}(E_j)$, is calculated for the case of Weibull loads as

$$g_{E,II}(E_j) = \left[\frac{1}{k}\left(\frac{\eta-\delta}{E_j}\right)^m\right] \times b^*_{DLDRII}\left(\frac{E_j-\delta}{\eta-\delta}, m, \beta, -\frac{\delta}{\eta-\delta}, Pk^Q\left(\frac{\eta-\delta}{E_j}\right)^{m_0}, m_0\right) \tag{A-4}$$

where the new class of generalized DLDR-based phase II b-functions is defined for $u \le \max(w_0, 0)$ as

$$b^*_{DLDRII}(w_0, m, \beta, u, y, m_0) = \int_{\max(w_0,0)}^{\infty} \frac{(w-u)^m}{1-\exp[y(w-u)^{m_0}]}\,\gamma(w,\beta)dw \tag{A-5}$$

It is noted that the DLDR phase I and phase II damage rates for Weibull loads, $g_{E,I}(E_j)$ and $g_{E,II}(E_j)$, respectively, may be substituted into equations used for various special cases in the APPLICATION NOTES section of the main body of this paper to calculate the effective composite DLDR damage rate, $g_E(E_j)$.